An approximate solution to the Boltzmann equation for vibrated granular disks

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The behaviour of the lower order moments of the velocity distribution function for a system of inelastic granular disks driven by vertical vibrations is studied using a kinetic theory. A perturbative kinetic theory for vibro-fluidised beds was proposed by Kumaran (JFM, v. 364, 163). A scheme to generalise this theory to higher orders in the moments is presented here. With such a method it is possible to obtain an analytical solution to the moments of the distribution function up to third order.

I. INTRODUCTION

The dynamics of vibrated granular materials, its instabilities, and pattern formation are of some interest in the recent years as demonstrated by experiments of [1] and simulations of [2]. The theoretical description of such systems is complicated by the fact that it is a driven dissipative system characterised by highly inelastic collisions and hence the validity of equations of hydrodynamics is not clear at present [3]. However, it is possible to describe one idealised situation, where the dissipation due to particle collisions is small and the amplitude of wall oscillations is small compared to the mean free path, as was shown in the kinetic theories [4, 5]. Such a description might be one of the starting points where we can ascertain with some confidence the rigour of the approach used. The present work is a continuation of such an approach.

In this communication, we show that it is possible to obtain an analytical solution to the Boltzmann equation for a dilute bed of vibrated granular disks, by the method of moments, correct upto third order in the moments of the distribution function. The method of approach followed holds the same principle as followed in [4], except in the choice of the distribution function, which is done here by expanding it in the orthogonal set of Hermite polynomials. It is hoped that the analysis of this base state solution for its stability would give some clues to understanding the instabilities occuring in a vibro fluidised bed.

In Sec. II, we present a general methodology to approximate the distribution function by expanding it in Hermite polynomials, and a procedure to solve the Boltzmann equation. We present only the important results here, the reader is referred to [4] for the details of the kinetic theory of vibro-fluidised beds. In Sec. III, we obtain an analytical solution when the formulation is restricted upto the third order in the moments. The results we obtain here are qualitatively similar to results of [4], except that we have obtained an analytical solution whereas in the latter it was an approximate series solution.

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II. BASIC FORMULATION

The Boltzmann equation for the velocity distribution function, $f(\mathbf{x}, \mathbf{u})$, for vertically vibrated beds is [4]:

$$\partial_t \hat{f} + u_i^* \partial_{i^*} \hat{f} - g \frac{\partial \hat{f}}{\partial u_z^*} = \frac{\partial_c \hat{f}}{\partial t} \tag{1}$$

where the collision integral is,

$$\frac{\partial_c \hat{f}}{\partial t} \equiv \sigma \int \mathrm{d}\mathbf{u}_2^* \mathrm{d}\mathbf{k} \; (\mathbf{w}^* \cdot \mathbf{k}) \left(\hat{f}_1' \hat{f}_2' - \hat{f}_1 \hat{f}_2 \right) \tag{2}$$

To obtain an approximate solution to this equation we expand the distribution function about a Maxwell distribution in some space as:

$$\hat{f}(\mathbf{x}, \mathbf{u}) = \frac{\rho}{T_0} f^0 [1 + A_j(\mathbf{x}) \varphi_j(\mathbf{u})] \tag{3}$$

where, $f^0(\mathbf{u}) \equiv \frac{1}{2\pi} \mathrm{e}^{u_i^{*2}/T_0}$ is the Maxwell distribution function with the T_0 left out of the definition for the simplifications to follow. The density ρ is also expanded about a leading order density field,

$$\rho = \rho_0 \left(1 + \rho_1\right) \tag{4}$$

The functions φ_j are chosen from a set of linearly independent function space. The parameters $A_j(\mathbf{x})$ are determined using the method of moments. In this method a set of functions, $\psi_i(\mathbf{u})$, equal in number to the number of unknowns are chosen and are multiplied with the Boltzmann equation, and integrated over the velocity space. This way we obtain a set of differential equations for the unknown parameters.

The leading order density and temperature distribution were obtained in [4] for a dilute bed.

$$\rho_0 = \frac{N\, g}{T_0} \, \mathrm{e}^{-g\, z/T_0} \tag{5}$$

$$T_0 = \frac{4\sqrt{2}}{\pi} \frac{\left\langle U^2 \right\rangle}{N\sigma(1-e^2)}. \tag{6}$$

Here, N is the number of particles per unit width of the bed, g is the gravitational accelaration, e is the coefficient of restitution for particle-particle collisions, and $\left\langle U^2 \right\rangle$ represents the mean square velocity of the vibrating surface. For sinusoidal forcing with characteristic velocity U_0, this is given by $\left\langle U^2 \right\rangle = U_0^2/2$.

The functions φ_j are chosen from a set of linearly independent function space. The parameters $A_j(\mathbf{x})$ are determined using the method of moments. In this method a set of functions, $\psi_i(\mathbf{u})$, equal in number to the number of unknowns are chosen and are multiplied with the Boltzman equation, and integrated over the velocity space. This way we obtain a set of differential equations for the unknown parameters.

a. Non-dimensionalisation As a simplification we use the following non-dimensionalisation. $\mathbf{u} = \mathbf{u}^*/\sqrt{T_0}$, $z = z^* g/T_0$. Substituting Eq. (3) in Eq. (1), multiplying by ψ_i and integrating over the velocity we obtain the steady state differential equation for the moments for variation only in the vertical direction, z as:

$$\frac{g}{T_0\sqrt{T_0}}\int \mathrm{d}\mathbf{u}_1\ \psi_i u_z f^0 \partial_z\ \rho(1 + A_j\varphi_j) + \frac{g\rho}{T_0\sqrt{T_0}}\int \mathrm{d}\mathbf{u}_1\ f^0(1 + A_j\varphi_j)\frac{\partial\psi_i}{\partial u_z} = \frac{\partial_c \hat{f}}{\partial t}. \quad (7)$$

Here the second term has been simplified using the divergence theorem and the condition that the distribution function vanishes for large velocities. When the collision term is integrated over the velocities, $\mathrm{d}\mathbf{u}_1$, it can be simplified from the form in Eq. (2) to an equivalent form (see in [6], for example)

$$\int \mathrm{d}\mathbf{u}_1\ \psi_i \frac{\partial_c \hat{f}}{\partial t} = \frac{1}{2}\sigma\sqrt{T_0}\int \mathrm{d}\mathbf{u}_1 \mathrm{d}\mathbf{u}_2 \mathrm{d}\mathbf{k}\ (\mathbf{w}\cdot\mathbf{k})\rho^2 \times f_1^0 f_2^0 (1 + A_j\varphi_{1j})(1 + A_k\varphi_{2k})\,\Delta\psi_i \quad (8)$$

where, $\Delta\psi_i = [\psi'_{1i} + \psi'_{2i} - \psi_{1i} - \psi_{2i}]$, is the total change in ψ_i due to collisions. The terms in Eq. (7) and Eq. (8) can be simplified by defining $\langle . \rangle \equiv \int \mathrm{d}\mathbf{u}\, f^0\, (.)$ and $\mathbb{C}[.]$ as the collision integral operator. Then we have from Eq. (7),

$$\begin{aligned} &\frac{g}{T_0\sqrt{T_0}}\left[\langle u_z\psi_i\rangle\,\partial_z\rho + \langle u_z\psi_i\varphi_j\rangle\,(A_j\partial_z\rho + \rho\partial_z A_j)\right] \\ &\quad + \frac{g\rho}{T_0\sqrt{T_0}}\left[\left\langle\frac{\partial\psi_i}{\partial u_z}\right\rangle + A_j\left\langle\frac{\partial\psi_i}{\partial u_z}\varphi_j\right\rangle\right] \\ &= \frac{\sigma\rho^2}{\sqrt{T_0}}\left[\mathbb{C}[\Delta\psi_i] + A_j\,\mathbb{C}[\Delta\psi_i\,\varphi_{1j}] + A_k\,\mathbb{C}[\Delta\psi_i\,\varphi_{2k}] + A_j A_k\,\mathbb{C}[\Delta\psi_i\,\varphi_{1j}\varphi_{2k}]\right] \end{aligned} \quad (9)$$

Further, dividing the above equation by $g/T_0\sqrt{T_0}$ and defining $\rho^* \equiv \ln\rho$, we obtain the following equation for the unknown variables.

$$S_i^0\partial_z\rho^* + S_{ij}(A_j\partial_z\rho^* + \partial_z A_j) + G_i^0 + G_{ij}A_j = \frac{\rho\,\sigma T_0}{g}\left(C_i^{0e} + C_i^{0i} + C_{ij}^{1e}A_j + C_{ijk}^{2e}A_jA_k\right) \quad (10)$$

where,

$$\begin{aligned} S_i^0 &= \langle u_z\psi_i\rangle \\ S_{ij} &= \langle u_z\psi_i\varphi_j\rangle \\ G_i^0 &= \left\langle\frac{\partial\psi_i}{\partial u_z}\right\rangle \\ G_{ij} &= \left\langle\frac{\partial\psi_i}{\partial u_z}\varphi_j\right\rangle \\ C_i^{0e} &= \mathbb{C}[\Delta\psi_i] \quad \text{with elastic collisions} \\ C_i^{0i} &= \mathbb{C}[\Delta\psi_i] \quad \text{inelastic collsions, excluding the above term} \\ C_{ij}^{1e} &= \mathbb{C}[\Delta\psi_i\,\varphi_{1j}] + \mathbb{C}[\Delta\psi_i\,\varphi_{2j}] \\ C_{ijk}^{2e} &= \mathbb{C}[\Delta\psi_i\,\varphi_{1j}\varphi_{2k}] \end{aligned}$$

Here, the superscript e for the collision terms above indicate that the collisions are considered to be elastic.

A leading order equation is obtained by considering the $A_j\varphi_j$ as perturbations to the Maxwell distribution and by considering elastic collision in the collision integral. Thereby we obtain by setting $A_j = 0$,

$$S_i^0\partial_z\rho^* + G_i^0 = \frac{\rho\sigma T_0}{g}C_i^{0e} \quad (11)$$

With, $\psi_i = u_z$, the leading order density variation is given by $\partial_z\rho^* = -1$, giving $\rho^* = -z + c$ or $\rho = \rho^{\mathrm{o}}\mathrm{e}^{-z}$, where $\rho^{\mathrm{o}} = Ng/T_0$ is the density at $z = 0$. Kumaran [4] had obtained the values of ρ^{o} and T_0 using a balance of the leading order source and dissipation, for low densities are given in Eqs. (5) and (6). A high density correction to these values was obtained in [5] in the leading order. In the present analysis we restrict ourselves to the low density limit. To obtain a first order balance in this limit, we neglect the quadratic term A_jA_k, and subtract out the leading order equation for low densities.

$$S_{ij}\partial_z A_j = \left[S_{ij} - G_{ij} + \frac{\mathrm{e}^{-z}}{\epsilon_l}C_{ij}^{1e}\right]A_j + \frac{\mathrm{e}^{-z}}{\epsilon_l}C^{0i} \quad (12)$$

Here we have used $1/\epsilon_l \equiv \rho^{\mathrm{o}}\sigma T_0/g = N\sigma$. Eqs. (12) is a set of coupled linear non-autonomous first order ordinary differential equations in the varibles A_j. If we incorporate the pertubation to density, Eq. (4), in Eq. (7), then the above equation for the first order quantities reads:

$$S_{ij}\partial_z A_j + S_i^0\partial_z\rho_1 = \left[S_{ij} - G_{ij} + \frac{\mathrm{e}^{-z}}{\epsilon_l}C_{ij}^{1e}\right]A_j + \left[S_i^0 - G_i^0 + \frac{2\,\mathrm{e}^{-z}}{\epsilon_l}C_{ij}^{0e}\right]\rho_1 + \frac{\mathrm{e}^{-z}}{\epsilon_l}C^{0i} \quad (13)$$

A. Boundary conditions

The boundary conditions for the above equations may be obtained by the following method. A balance of the value of a moment ϕ_i is considered when it collides with a wall moving with a velocity U. The change in the value of a moment due

to the collision is given by the relation:

$$\int_{U}^{\infty} \mathrm{d}u_z^* \int_{-\infty}^{\infty} \mathrm{d}u_x^*\, f(\mathbf{u})\phi_i(\mathbf{u}) = \int_{-\infty}^{U} \mathrm{d}u_z^* \int_{-\infty}^{\infty} \mathrm{d}u_x^*\, f(\mathbf{u})\phi_i(\mathbf{u}) + \int_{-\infty}^{U} \mathrm{d}u_z^* \int_{-\infty}^{\infty} \mathrm{d}u_x^*\, f(\mathbf{u})\Big[\phi_i(\mathbf{u}') - \phi_i(\mathbf{u})\Big] \quad (14)$$

where the primed variable deotes the velocity of the particle after a collision. Simplifying the above equation we get,

$$\int_{-\infty}^{U} \mathrm{d}u_z^* \int_{-\infty}^{\infty} \mathrm{d}u_x^*\, f(\mathbf{u})\phi_i(\mathbf{u}') = \int_{U}^{\infty} \mathrm{d}u_z^* \int_{-\infty}^{\infty} \mathrm{d}u_x^*\, f(\mathbf{u})\phi_i(\mathbf{u}) \quad (15)$$

If the mean free path of the particles is large compared to the amplitude of vibration, then we can make an assumption that the wall is stationary at one position, then the above equation can be further simplified by averaging over different probable velocities of the bottom wall, which, in the case of a sine wave oscillation is,

$$P(U)\mathrm{d}U = \frac{1}{\pi(U_0^2 - U^2)} \quad (16)$$

if we write $U = U_0 \sin\theta$, and $\epsilon \equiv U_0^2/T_0$, then the averaged equation in terms of the nondimensional quantities for the balance of ϕ_i, after making the substitution for the distribution function at $z = 0$, will be,

$$A_j \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_{-\infty}^{\sqrt{\epsilon}\sin\theta} \mathrm{d}u_z \int_{-\infty}^{\infty} \mathrm{d}u_x\, f^0 \phi_i' \varphi_j - A_j \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_{\sqrt{\epsilon}\sin\theta}^{\infty} \mathrm{d}u_z \int_{-\infty}^{\infty} \mathrm{d}u_x\, f^0 \phi_i \varphi_j = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_{\sqrt{\epsilon}\sin\theta}^{\infty} \mathrm{d}u_z \int_{-\infty}^{\infty} \mathrm{d}u_x\, f^0 \phi_i - \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_{\infty}^{\sqrt{\epsilon}\sin\theta} \mathrm{d}u_z \int_{-\infty}^{\infty} \mathrm{d}u_x\, f^0 \phi_i' \quad (17)$$

or simply as,

$$E_{ij} A_j(0) = B_i \quad (18)$$

Solving these simultaneous equations we obtain conditions satisfied by A_j at $z = 0$.

III. THIRD-ORDER MOMENT FORMULATION

We now take the specific case of the third order formulation. A third order formulation is the lowest order in which we can include the anisotropy in the distribution function. We now briefly explain the choice of the moments φ_j. The distribution function satisfies the following criteria. (i) the time averaged vertical flux is zero, $\langle u_z \rangle = 0$, (ii) the distribution function is normalised to unity, (iii) it is symmetric in the horizontal velocities. It is convenient to choose the lower order moments from a multi-dimensional Hermite polynomials for the following reasons. These polynomials form a linearly independent orthogonal basis, and the resulting equations are more convenient than a linearly independent set. In addition two of the above conditions will be automatically satisfied by the distribution function by setting the corresponding the corresponding A_j to zero. Note that since the leading order distribution function f^0 already satisfies the unit normalisation condition, we would require that $A_j \langle \varphi_j \rangle = 0$. The flux condition requires that $A_j \langle u_z \varphi_j \rangle = 0$. Since 1 and u_z are two of the linearly independent functions in the orthogonal Hermite polynomials, setting the coefficients of these two will ensure that the above two conditions will be satisfied at all orders of the polynomials. Thus the choice of the φ_j becomes simple by incorporating the constraints on the distribution function directly.

The set of multi-dimensional orthogonal polynomials can be obtained from the recurrence relation for the Hermite polynomials,

$$H^{n+1}(x) = xH^n(x) - nH^{n-1}(x) \quad (19)$$
$$\text{with} \quad H^0(x) = 1, \quad H^1(x) = x$$

and the multidimensional set is then given by,

$$H(u_x, u_z) = H^m(u_x)H^n(u_z) \quad \text{even } m, \text{ all } n. \quad (20)$$

A symmetric distribution in the horizontal velocity can be ensured by taking only even powers of u_x, ie., even m in the above expression. The above polynomials are can be normalised to unity and the factor is $1/m!\,n!$.

In the case of the third order approximation we obtain the

following polynomials,

$$\varphi_j = \{-1+u_z^2, -3u_z+u_z^3, -1+u_x^2, -u_z+u_x^2 u_z\}, \tag{21}$$

in which the first two members viz., 1 and u_z have been omitted for reasons discussed above. We choose the same set for the moment generating functions ψ_i in Eq. (13) and the functions for the boundary conditions, ϕ_i in Eq. (18). This way we can get the same order of representation in the moment equations as well as in the boundary conditions.

The moments of the distribution can be obtained by substituting the expression (21) in Eq. (3), which are:

$$\left\langle u_x^2 \right\rangle = (1+2A_3) \tag{22a}$$
$$\left\langle u_z^2 \right\rangle = (1+2A_1) \tag{22b}$$
$$\left\langle u_x^2 u_z \right\rangle = 2\,A_4 \tag{22c}$$
$$\left\langle u_z^3 \right\rangle = 6\,A_2 \tag{22d}$$

IV. SOLUTION

With the moments considered in Eq. (21), the set of relations in Eq. (12) for A_i can be written as

$$\partial_z A_i = L^0_{ij} A_j + L^1_{ij} A_j \mathrm{e}^{-z} + b^1_i \mathrm{e}^{-z}, \tag{23}$$

where,

$$L^0_{ij} = S^{-1}_{ik}(S_{kj} - G_{kj}), \quad L^1_{ij} = \frac{1}{\epsilon_l} S^{-1}_{ik} C^{1e}_{kj}, \quad b^1_i = \frac{1}{\epsilon_l} S^{-1}_{ik} C^{0i}_k,$$

and by considering a moment $\psi_i = u_z$ in Eq. (13) we have for the density correction,

$$\partial_z \rho_1 + 2\,\partial_z A_1 = 2\,A_1, \tag{24}$$

In the case of the third order approximation, we have

$$\begin{aligned} L^0_{ij} &= \{\{0,0,0,0\},\{0,1,0,0\},\{0,0,0,0\},\{0,0,0,1\}\} \\ L^1_{ij} &= \tfrac{\sqrt{\pi}}{\epsilon_l}\{\{0,-\tfrac{3}{2},0,\tfrac{1}{2}\},\{-\tfrac{1}{3},0,\tfrac{1}{3},0\},\{0,\tfrac{3}{2},0,-\tfrac{5}{2}\}, \\ &\quad \{1,0,-1,0\}\} \\ b^1_i &= -\tfrac{(1-e^2)\sqrt{\pi}}{6\epsilon_l}\{0,1,0,3\} \end{aligned}$$

We note here that the A_i are independent of the density correction ρ_1 in the first order approximation These equations can be rearranged into a single fourth order equation in A_1, which is easily accomplished through a symbolic routine:

$$A_1^{(4)}(z) = -4\,A_1^{(3)}(z) + \left(\frac{4\pi}{\epsilon_l^2}\mathrm{e}^{-2z} - 4\right) A_1''(z) - \frac{\pi^2}{\epsilon_l^4}(1-e^2)\,\mathrm{e}^{-4z} \tag{25}$$

With this simplification, the other variables can be written down in terms of A_1 and its derivatives as

$$\begin{aligned} A_2(z) = {}& \frac{(-1+e^2)}{24}\left(\tfrac{\epsilon_l}{\sqrt{\pi}}\,\mathrm{e}^z\right)^{-1} \\ &+ \left(-A_1'(z) + \frac{A_1''(z)}{6}\right)\left(\tfrac{\epsilon_l}{\sqrt{\pi}}\,\mathrm{e}^z\right) \\ &+ \left(\frac{A_1''(z)}{6} - \frac{A_1^{(4)}(z)}{24}\right)\left(\tfrac{\epsilon_l}{\sqrt{\pi}}\,\mathrm{e}^z\right)^3 \end{aligned} \tag{26a}$$

$$A_3(z) = A_1(z) - A_1''(z)\left(\tfrac{\epsilon_l}{\sqrt{\pi}}\,\mathrm{e}^z\right)^2 \tag{26b}$$

$$\begin{aligned} A_4(z) = {}& \frac{(-1+e^2)}{8}\left(\tfrac{\epsilon_l}{\sqrt{\pi}}\,\mathrm{e}^z\right)^{-1} \\ &+ \left(-A_1'(z) + \frac{A_1''(z)}{2}\right)\left(\tfrac{\epsilon_l}{\sqrt{\pi}}\,\mathrm{e}^z\right) \\ &+ \frac{1}{2}\left(A_1''(z) - \frac{A_1^{(4)}(z)}{4}\right)\left(\tfrac{\epsilon_l}{\sqrt{\pi}}\,\mathrm{e}^z\right)^3 \end{aligned} \tag{26c}$$

To solve Eq. (25), we can make a reduction in order by the following substitution

$$A''(z) = y(z) \tag{27}$$

With the transformation $x = \mathrm{e}^{-z}$, we obtain from Eq. (25), (Note: here x is just a transformation variable and not the cartesian co-ordinate)

$$\ddot{y} - 3\frac{\dot{y}}{x} - \frac{1}{x^2}(c\,x^2 - 4)\,y = D\,x^2 \tag{28}$$

where a dot accent denotes ∂_x, and the constants are $c \equiv 4\pi/\epsilon_l^2$, $D \equiv -\pi(1-e^2)/\epsilon_l^4$. Further with

$$w(x) = y(x)/x^2, \tag{29}$$

we get,

$$\ddot{w} + \frac{\dot{w}}{x} - c\,w = D, \tag{30}$$

which is a modified Bessel equation of zeroth order. The solution to the homogeneous equation is given by:

$$w_h(x) = c_1\,I_0(\sqrt{c}\,x) + c_2\,K_0(\sqrt{c}\,x) \tag{31}$$

A particular solution to the inhomogeneous equation can be obtained by variation of parameters. The Wronskian of the above solutions can be written down from standard references,

$$\begin{vmatrix} I_0(\sqrt{c}\,x) & K_0(\sqrt{c}\,x) \\ \dot{I}_0(\sqrt{c}\,x) & \dot{K}_0(\sqrt{c}\,x) \end{vmatrix} = -\frac{1}{x}. \tag{32}$$

The particular solution so obtained is given by:

$$w_p(x) = -\frac{D\,x}{\sqrt{c}}\,\left[I_0(\sqrt{c}\,x)\,K_1(\sqrt{c}\,x) + K_0(\sqrt{c}\,x)\,I_1(\sqrt{c}\,x)\right]. \tag{33}$$

The most general solution to the inhomogeneous equation is then

$$w(x) = w_h + w_p(x). \tag{34}$$

The Eq. (27) can now be solved in terms of the independent variable x, by further reduction in order, by writing it,

$$x\left(x\ddot{A}_1(x) + \dot{A}_1(x)\right) = y(x), \tag{35}$$

as:

$$\dot{B}(x) = w(x) - \frac{B(x)}{x} \tag{36}$$

where,

$$\dot{A}_1(x) = B(x). \tag{37}$$

The solutions to these first order equations can be easily written as

$$B(x) = \frac{1}{x}\left[\int^{x}\mathrm{d}x'\, w(x')\, x' \;\; + \;\; c_3\right] \tag{38}$$

and

$$A_1(x) = \int^{x}\mathrm{d}x''\,\frac{1}{x''}\left[\int^{x''}\mathrm{d}x'\, w(x')\, x' \;\; + \;\; c_3\right] \;\; + \;\; c_4. \tag{39}$$

The correction to the density is then from Eq. (24),

$$\rho_1(x) = -2\int^{x}\mathrm{d}x'\,\left(\frac{A_1(x')}{x'} + B(x')\right) + c_5. \tag{40}$$

The derivatives of $A_1(z)$ can be written down as

$$A_1'(z) = -x\,\dot{A}_1(x) = -x\,B(x) \tag{41a}$$
$$A_1''(z) = -x^2\,w(x) \tag{41b}$$
$$A_1'''(z) = -2x^2\,w(x) - x^3\,\dot{w}(x) \tag{41c}$$

These are used to evaluate the other unknown functions in Eqs. (26).

A. Boundary conditions

Some of the constants of integration can be directly eliminated by requiring that the functions to take finite values for $x = 0$ (or equivalently, $z \to \infty$). Consider one of the linearly independent solutions of $w_h(x)$ in Eq. (31), $c_2\,K_0(\sqrt{c}\,x)$, which when integrated through Eqs. (38) and (39) gives a term in $A_1(x)$: $c_2\,K_0(\sqrt{c}\,x)/c$. Since $K_0(\sqrt{c}\,x)$ is singular as $x \to 0$, we require $c_2 = 0$. Similarly we require that $c_3 = 0$, as this will lead to a singular term in $A_1(x)$: $c_3\,\ln x$.

The expressions in Eq. (33) cannot be integrated to obtain a closed form expression. Nevertheless, they can be easily evaluated numerically by converting the expressions Eq. (38) and (39) to a definite integral, $\int_{x_0}^{x}\mathrm{d}x'$ added with some constant. The choice of the lower limit, x_0, can, in general, be artibary to suit the convinience of matching the given boundary conditions; but such a choice, say $\int_{1}^{x}\mathrm{d}x'$, for the integral in Eq. (38) will lead to singularities in the expression for $A_1(x)$ in Eq. (39), similar to those obtained by the constant c_3 term. To do away with this singularity, we choose $x_0 = 0$, expand the integral in Eq. (38) in Taylor's series, and subtract out the source of the singularity (which essentially is the constant of integration, *i.e.*, the value of the integral evaluated at the lower limit). In expanding $w(x)$ about $x = 0$, we note from Eq. (34) that $w(0) = 0$, therefore,

$$\begin{aligned} B(x) = \frac{1}{x}\int_{0}^{x}\mathrm{d}x'\, w(x')\, x' &= \frac{1}{x}\int_{0}^{x}\mathrm{d}x'\,\left(w_1 x' + w_2 x'^2 + \ldots\right)\, x' \\ &= \left(\frac{w_1}{2}x + \frac{w_2}{3}x^2 + \ldots\right), \end{aligned}$$

where $w_1, w_2, \ldots$ are constants coming from the Taylor's expansion. Such an expansion is possible as the series converges for $0 \le x < 1$. The above expression identically vanishes at $x = 0$, therefore we donot have to explicitly subtract out any singularity during numerical computation of the definite integral. Such problems of subtracting out the singularity, however, does not arise in the integral of Eq. (39) and any arbitary point, x_0, can simply be chosen for the numerical integration, keeping in mind the boundary conditions.

We are now left with two arbitary constants, c_1 and c_4. These are evaluated using boundary conditions in a manner similar to those used in [4]. For the sake of convinience we choose the lower limit for the definite integral in Eq. (39) to be $x_0 = 1$, then c_4 will simply be equal to the value of $A_1(x = 1)$. It can be seen from Eq. (22b) that A_1 is directly propotional to the moment of the distribution function $\langle u_z^2\rangle$, whose value at $z = 0$ can be obtained by considering it as a vertical flux of momentum because of collisions with the wall. Assuming that the particles collide with the wall have a leading distribution to be a Maxwellian, the flux of momentum along the vertical direction is given by,

$$\left.\langle u_z^2\rangle\right|_{z=0} = 1 + \frac{\epsilon}{2} \tag{42}$$

substituting in Eq. (22b) we have, $\left.A_1\right|_{(x=1)} = \epsilon/4$, therefore,

$$c_4 = \frac{\epsilon}{4}. \tag{43}$$

Since, momentum is transferred only in the vertical direction we have from Eqs. (22), $A_3|_{x=1} = 0$ and $A_4|_{x=1} = 0$. The constant, c_1, can now be evaluated from Eqs. (27), (29) and (34).

$$c_1 = \frac{1}{I_0(\sqrt{c})}\;\left.\left(A_1''(z) - w_p(x)\right)\right|_{z=0 \text{ or } x=1}, \tag{44}$$

The constant $\left.A_1''(z)\right|_{z=0}$ can be solved for from Eqs. (26)

$$\begin{aligned} \left.A_1''(z)\right|_{z=0} &= \frac{\pi}{\epsilon_l^2}\;\left.(A_1 + A_3)\right|_{z=0} \\ &= \frac{\pi\,\epsilon}{4\,\epsilon_l^2} \end{aligned}$$

The other constant c_5, is obtained by considering mass balance for the density correction ρ_1. Since the total mass of the bed is balanced in the leading order density profile ρ_0, the balance for the correction to the density is given by

$$\int_0^\infty \mathrm{d}z^* \, \rho_0 \rho_1 = 0, \tag{45}$$

or,

$$\int_0^1 \mathrm{d}x \, \rho_1 = 0. \tag{46}$$

We note here that we have not strictly incorporated the boundary conditions as discussed in Sec. II. This is because of a need to satisfy more strong boundary conditions of non-diverging solutions for $z \to \infty$. The equations of the sort of Eq. (18) would be useful in obtaining, for example, a series solution by numerical methods where we expand the solution in decaying functions such as Laguerre polynomials.

Furthermore, the moments related to the horizontal direction are not exactly satisfied, *i.e.*, the boundary conditions related to A_3 and A_4. While the value of A_3 is not satisfied independently, but only partly in Eq. (44), A_4 is never used. These can be used only in a higher order polynomial approximation. Strictly therefore, only A_1 is satisfied exactly.

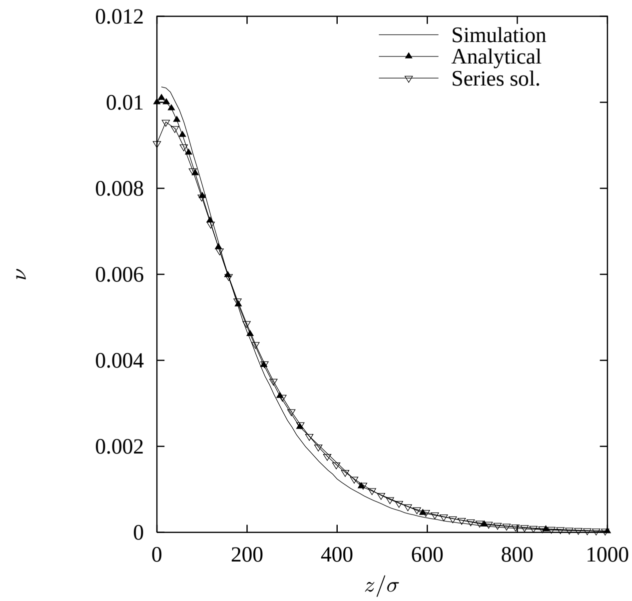

FIG. 1: Density profile for $N\sigma = 3$, $\epsilon = 0.3$. The analytical solution obtained in this paper and the series solution of [4] show good agreement. There is a negative correction to the density at the bottom after which it decays exponentially as the leading order values.

V. DISCUSSION

The results obtained here are qualitatively the same as obtained in [4]. At the time of the previous work, the only results available were some experimental measurements of [7] and only a qualitative comparison was possible to ascertain the validity of the theory. We have now compared the results of with a numerical Event Driven (ED) simulation of vibrated hard disks. The ED simulation was done with periodic boundaries and with an approximate representation of the bottom wall [8]. The following figures show that the theory is indeed good in the limit of its validity. The figures also show a comparison with the approximate series solution which was obtained in [4]. In most cases the series solution provides a good approximate.

The parameters chosen for the simulation are in correspondence with the limit of validity of the theory: $\epsilon \ll 1$, and $N\sigma \sim \mathcal{O}(1)$, (see [5, 8], for a discussion). There is a small negative correction to the density near the bottom wall, as shown in Fig. 1, due to the energy flow near the bottom wall, after which it falls of exponentially as the leading order density.

One important difference, in the formulation of the density, between the present and the previous work [4] is the following. In the previous work a correction to the distribution function due to variations in the distribution function over distances comparable to the particle diameter were described by using a small parameter, ϵ_G. It was shown in a later work [8] that this correction is essentially equivalent to the high density correction obtained on the lines of Enskog correction to dense gases, therefore it has been omitted in the present consideration of dilute bed.

Due to the anisotropic nature of the energy input to the vibro-fluidised bed, the lower order moments of the distribution clearly show the anisotropy. Figs. 2 and 3 show the horizontal and vertical temperature profiles, while Fig. 4 shows the vertical flow of energy. Anisotropies were also observed by us in deep bed simulations of disks which had the wave-like surface patterns [9]; although the nature of anisotropy was more pronounced even in the shape of the distribution function itself (the vertical distribution had double peaks and the horizontal distribution had single peak and exponential tails). Could the presence of anisotropy be an important feature giving rise to an instability in one direction ? A stability analysis of the solution from the present analysis model might help resolve this. The usual models based on hydrodynamic equations do not take into account this anisotropy.

As pointed out earlier, the density correction considered here is different from the one used in the previous work, in that the high density correction is not considered in first order in the distribution function. This results in a better prediction of the vertical temperature Fig. 5 and flux of energy Fig. 6 even when the density prediction is not expected to be good.

To conclude, we have shown that (a) It is possible to obtain an analytical solution to the Boltzmann equation for vibro fluidised bed in the low density limit correct upto the third order in the moments of the distribution function. (b) The qualitative nature of results obtained here are similar to those obtained in [4]. (c) The correction to the distribution function

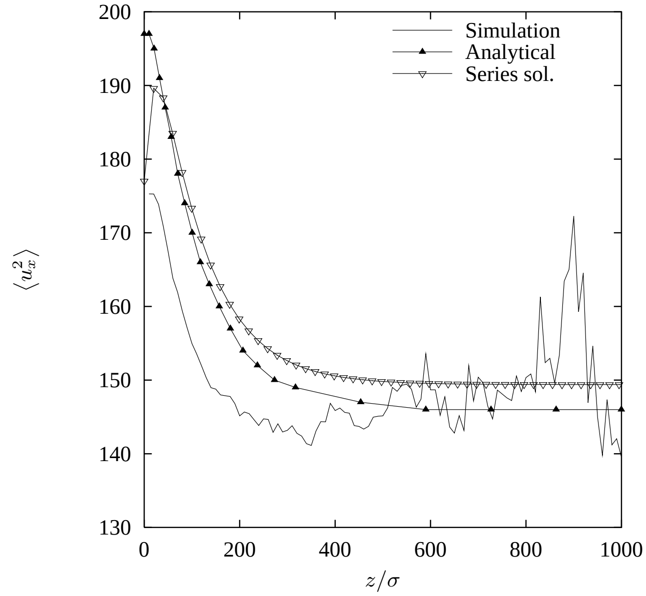

FIG. 2: Horizontal temperature profile for $N\sigma = 3$, $\epsilon = 0.3$. A magnitude comparison with Fig. 3 shows the degree of anisotropy in a vibrated bed.

due to spatial variation of the order of a particle diameter was neglected here as this is turns out to be a a correction to a higher order in density [8]. With this it is seen that the density still shows a negative correction at the bottom wall due to the energy flow. (d) The boundary conditions are overspecified in the problem and we chose to satisfy exactly, only the ones involved in the momentum transfer the vertical direction. (e) Even with this restricted choice, the theoretical values for the different moments compare reasonably well with the simulation particularly in the anisotropy exhibited. (f) This gives us some confidence to explore further with the stability of the solution, and with an higher order approximation to the distribution function if required.

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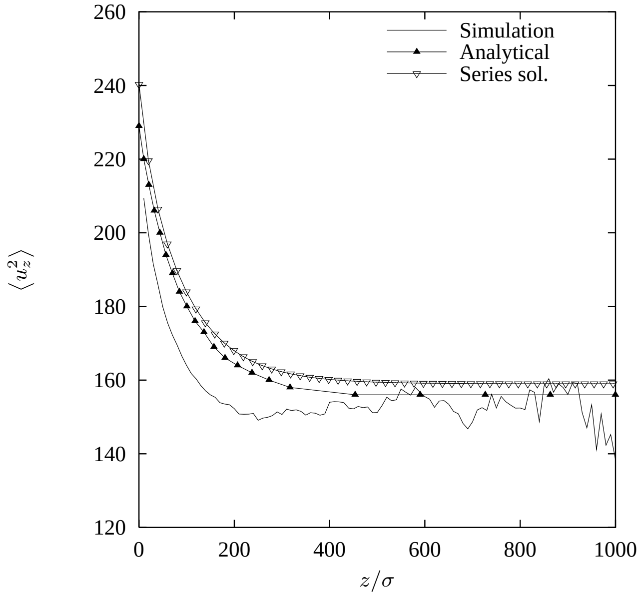

FIG. 3: Vertical temperature profile for $N\sigma = 3$, $\epsilon = 0.3$. The theoretical values are closer to the simulation here, than for the horizontal temperature because of the boundary conditions imposed.

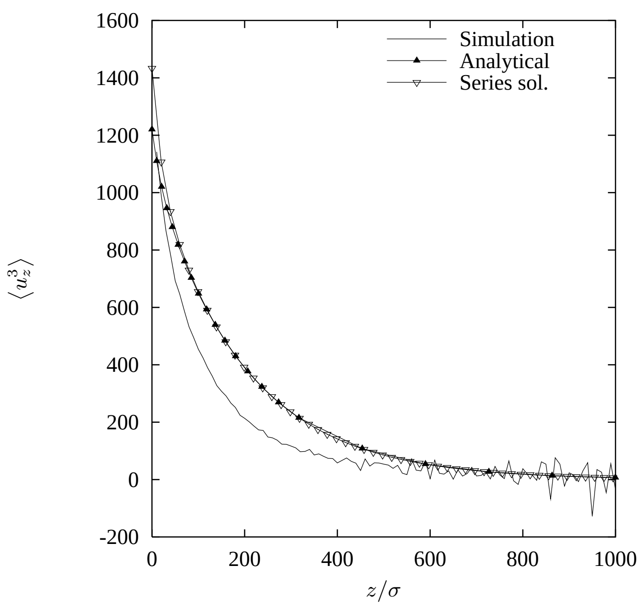

FIG. 4: Flux of energy in the vertical direction for $N\sigma = 3$, $\epsilon = 0.3$. Here again, the theoretical values compare well in order of magnitude, because of boundary conditions are imposed in this direction.

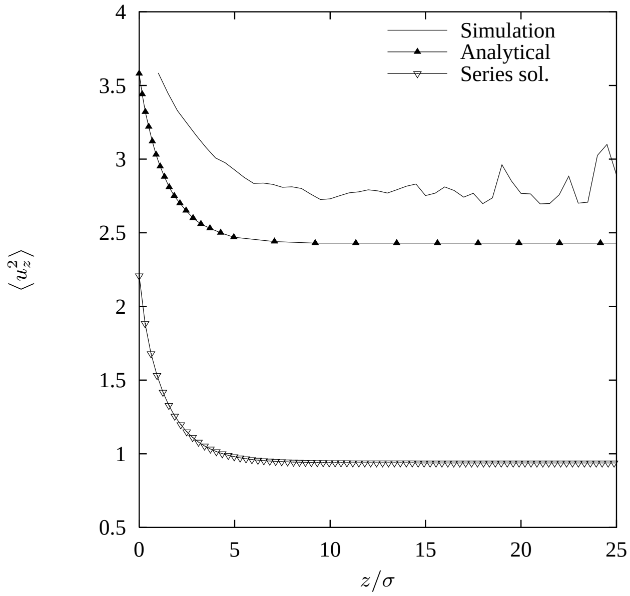

FIG. 5: Vertical temperature profile prediction is good even when the density is high ($\nu \sim 0.3i$).

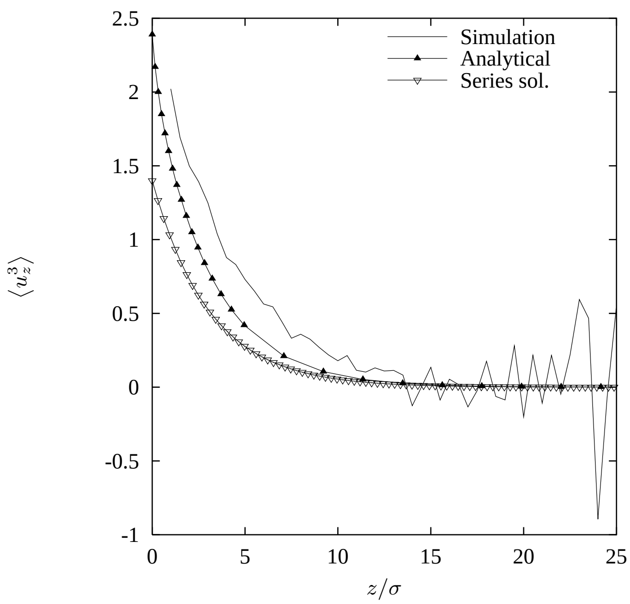

FIG. 6: Flux of energy prediction is good even when the density is high ($\nu \sim 0.3$).